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## **A Re-emergent Analysis of Early Algebraic Learning**

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**Abstract:** In this paper, we discuss a novel approach for collaborative retrospective analysis. One researcher was directly involved in a classroom teaching experiment, adopting an emergent perspective as an interpreter-witness of classroom interactions during a four-week algebra instructional unit with sixth-grade students. The other researcher experienced and analyzed the data in reverse chronological order. We describe how this re-emergent perspective revealed aspects of students' early algebraic reasoning.

**Keywords:** early algebra, emergent perspective, constructivism, design research

### **Introduction**

“Hindsight is twenty-twenty” and “eyes in the back of your head” are two phrases that are distinct but related in meaning. The former refers to the revealing of prior misapprehension, whereas the latter refers to the ability to see what seems imperceptible. In mathematics education research, we often video-record teaching sessions so we can attempt to analyze teaching and learning with “hindsight in the back of our heads.” We slow down recordings to identify previously ambiguous nuance; we analyze and re-analyze to consider changes in what we notice transpiring between sessions and retrospectively across sessions.

Another common phrase relevant to analyses of teaching experiments is “two heads are better than one.” This manifests in the need for a witness of the teaching sessions (Steffe & Thompson, 2000) to assist with in-the-moment inferences and both on-going and *retrospective*

analyses. There are many approaches to retrospective analyses of teaching experiments in mathematics education research. Some researchers focus on describing characteristics of the product of analysis, for instance, a stability in a researcher's model of a students' mathematics, without prescribing the process explicitly (Steffe & Thompson, 2000). Others (e.g., Hackenberg & Lee, 2016; Hunt, Tzur, & Westenskow, 2016) reference aspects of grounded theory methodology, such as the constant comparative method (Corbin & Strauss, 2008). In this paper, we describe a new perspective for retrospective collaborative analysis, what we term a "re-emergent perspective".

One of us (Moss) was directly involved throughout a design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) investigating a hypothetical learning trajectory (HLT) for the sixth-grade objectives of the "Expressions and Equations" strand of the United States' Common Core State Standards for School Mathematics (Figure 1, NGA/CCSSO, 2010). Moss had adopted an *emergent* perspective as an interpreter-witness of classroom interactions (Cobb & Yackel, 1996). Moss designed activities and lessons and regularly met with the classroom teacher between teaching sessions. The other author (Boyce) was not involved at all with the experiment until after data collection was complete. Boyce formed hypothetical models of four students' cognitive activities as classroom activities were revealed in *reverse* chronological order, via video recordings. Boyce was situated in what we term a *re-emergent* perspective. Boyce formed conjectures about relationships between instruction and students' mathematics, and he revised those conjectures as previous sessions were revealed. A value we see in re-emergent analysis is that it allows researchers with different but complementary theoretical perspectives to compare interpretations of the outcomes of a design experiment.

### **Relationship with the Emergent Perspective**

We were motivated to collaborate when we became aware of one another's research interests in an informal setting. During a mentorship program for junior mathematics education faculty (the Association of Mathematics Teacher Educators' STaR Fellowship Program, <http://amte.net/star>) we learned that we shared an interest in researching sixth-graders algebraic reasoning. Boyce had been immersed in research from small-group teaching experiments, primarily based in Les Steffe's work, that suggested relationships between students' understandings of algebraic expressions and their understandings of fractions (e.g., Hackenberg & Lee, 2015). Meanwhile, Moss had conducted a classroom teaching experiment from which she developed a learning trajectory for sixth-grade students' reasoning about variables and equations (Moss, 2014).

In relation to Cobb, Stephan, McClain, and Gravemeijer's (2001) interpretive framework (see Figure 1), our shared focus was the last row: relationships between classroom mathematical practices and mathematical interpretations and reasoning. We were both interested and immersed in literature pertaining to middle grade students' learning of (early) algebra, and we had research experiences and expertise that were complementary. Moss had adopted a social perspective in her study, focusing on classroom norms, "taken-as-shared" meanings, and participation in mathematical discourse (Cobb, 1999). Boyce adopted a psychological perspective in analyzing Moss' data, focusing on students' cognitive schemes and analyzing whether their understandings were procedural, participatory, or anticipatory (Tzur & Simon, 2004; von Glasersfeld, 1995).

We believe other researchers with compatible interests within the interpretative framework might benefit from collaboration using the re-emergent perspective, depending on the specificity of their shared interest and expertise. In the next sections, we describe the design

experiment and results of analyses from our two perspectives. As we will discuss, a main constraint on analyses from the re-emergent perspective are the qualities of data collected for unanticipated analyses.

| <b>Social Perspective</b>        | <b>Psychological Perspective</b>   |
|----------------------------------|--|
| Classroom social norms           | Beliefs about own role, others' roles, and the general nature of mathematical activity in school |
| Sociomathematical norms          | Mathematical beliefs and values  |
| Classroom mathematical practices | Mathematical interpretations and reasoning   |

Figure 1. Interpretive Framework for analyzing communal and individual mathematical activity and learning (Cobb et al., 2001).

### About the Design Experiment

Preliminary topics in algebra usually consist of variables, simplification of algebraic expressions, equations in one unknown, and equation solving (Kieran, 1989). Although school algebra often places emphasis on manipulations of variables and symbols, algebra is more than a set of procedures for manipulating symbols (NCTM, 2000). Furthermore, according to Kirshner (1993), a drill approach to symbol manipulation is undesirable because it “trains students in non-reflective competence” (p. 3). Part of the intent of early algebra is to understand and leverage ways elementary and middle grade students’ numerical and quantitative reasoning relate to their learning of algebraic concepts (Brizuela & Schliemann, 2004; Empson, Levi, & Carpenter, 2011; Hackenberg, 2013; Hackenberg & Lee, 2015; Kaput, 1999; NCTM, 2000).

The United States’ Common Core State Standards for School Mathematics (CCSS-M) stipulate that students should begin to develop more formal understandings of variables, expressions, and equations in sixth grade (NGA/CCSSO, 2010). Moss (2014) conducted a

whole-class design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) in a U. S. sixth grade classroom to investigate the teaching and learning of the sixth-grade objectives of the “Expressions and Equations” strand of the CCSS-M standards (Figure 2). Moss’ research focused on the development of a hypothetical learning trajectory (HLT) (Simon, 1995) for these mathematical concepts. An HLT consists of three inter-related components: goals for mathematics learning, tasks designed to promote learning, and hypotheses about the process of students’ learning. As part of her documenting and analyzing classroom learning, Moss participated as an observer and co-teacher in the classroom, which was lead-taught by an experienced teacher.

- Apply and extend previous understandings of arithmetic to algebraic expressions.
- Reason about and solve one-variable equations and inequalities.
- Represent and analyze quantitative relationships between dependent and independent variables.
- Use properties of operations to generate equivalent expressions.
- Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

Figure 2. Sixth grade expressions and equations strand (NGA/CCSSO, 2010)

There were 22 students in the sixth-grade class, and the teaching experiment consisted of 20 hour-long sessions over four consecutive weeks at the very beginning of the school year. Students sat in groups of three to four and participated in small group and whole class discussions. During each discussion, a few students shared their work and mathematical reasoning with the whole class. The students used a document camera and the board at the front of the classroom to share their thinking. While students presented, their peers in the class asked

questions and compared their reasoning to the presenters' thinking, which generated whole class discussion. The teacher facilitated the lessons and encouraged students to come to a shared understanding of the mathematics being discussed in each lesson. Students also engaged in small group discussions in which they could share their work with each other and help each other to clarify their reasoning.

## Analyses from the Emergent Perspective

### Prospective analyses

Prospective analyses refer to ongoing, between-session research that is the work of lesson planning. In the design experiment, lessons were modified and reorganized daily based on Moss' understandings of relationships between the students' learning and the teacher's implementation of tasks. Moss logged her observations documenting each teaching session in terms of mathematical meanings, errors, activities, discussions, teacher meanings, and justifications for modifications in the lessons. Figure 3 shows an example of a lesson log for the teaching session on adding and subtracting like terms.

|                                   |  |
|-----------------------------------|--|
| <b>Date/<br/>Activity</b>         | <b>Day 5</b><br><b>9/13/13</b><br><b>Adding and Subtracting Like Terms</b><br>Students will develop an understanding of variables in mathematics and will learn that like terms can be added and subtracted. Students will also learn to model patterns with algebra.  |
| <b>Mathematical<br/>Meaning</b>   | <ul style="list-style-type: none"> <li>• Simplifying</li> <li>• Like Terms</li> <li>• Unlike Terms</li> <li>• Expression</li> <li>• Equation</li> </ul>  |
| <b>Errors/<br/>Misconceptions</b> | <ul style="list-style-type: none"> <li>• Combining like terms: <math>3a + 4a + 7b = 14f</math><br/>e.g., 3 apples + 4 apples + 7 bananas = 14 fruits</li> <li>• Subtraction of like terms: All like terms are added.</li> <li>• Like Terms: Writing the variable in front of the coefficient.<br/>e.g., <math>R3 + B2 + G7</math> instead of <math>3R + 2B + 7G</math></li> <li>• Like term is correct and unlike term is wrong.</li> <li>• Like terms have the same coefficient and same variable<br/>e.g., <math>20h</math> and <math>20h</math> are like terms, but <math>5h</math> and <math>15h</math> are not like terms.</li> </ul> |
| <b>Activity that led to</b>       | <ul style="list-style-type: none"> <li>• More Practice (Day 5) worksheet.</li> </ul>   |

|   |   |
|---|---|
| <b>misconception</b>                        | <ul style="list-style-type: none"> <li>Identifying the like terms and the unlike terms and then writing an algebraic expression for each situation.</li> </ul>  |
| <b>Context of small groups</b>              | <ul style="list-style-type: none"> <li>Discussion about adding and subtracting like variables.</li> <li>Small groups were able to write the expression, but not simplify the expression.</li> </ul>   |
| <b>Context of whole group</b>               | <ul style="list-style-type: none"> <li>Discussion about equality and why both sides of an equation have to be the same.</li> <li>Simplifying the expression to an equivalent expression.</li> <li>Used an arrow (<math>\rightarrow</math>) to show simplification instead of an equal sign (<math>=</math>)</li> </ul>  |
| <b>Role of teacher/ teacher conceptions</b> | <ul style="list-style-type: none"> <li>The lesson went long (2 hours).</li> <li>Too much information for them.</li> <li>Simplifying is a new idea, so that is why she used the arrow to show simplifying first.</li> <li>Teacher thinks it is better to start with simplifying abstract problems and then go to context problems.</li> <li>Need more review of equal sign.</li> <li>The teacher led the whole class discussion and helped students come up with the equal sign means the same on both sides.</li> </ul> |
| <b>What did we change and why?</b>          | <ul style="list-style-type: none"> <li>Simplifying was not originally part of this lesson. Introduced the idea of simplifying. (<i>I think the arrow is going to confuse them/ need to use equal sign</i>)</li> <li>Did not get to the cost of a soccer ball.</li> <li>Spent more time on simplifying and equality than expected.</li> </ul>  |

Figure 3. Example of daily log content

Figure 3 refers to a lesson involving the context of adding and subtracting apples and bananas. It is important to note that at this stage in the instructional unit, many students were using a letter to label an object instead of a quantity of object, for example,  $3a$  is 3 apples. Additionally, students wrote the “variable” in front of the coefficient (e.g.  $R3$  instead of  $3R$ ). Also, during this lesson the teacher began using an arrow to show simplification instead of the equal sign. Based on these observations and others (shown in Figure 3), the next lesson began with a whole class discussion with the objective of clarifying the misconceptions about combining like terms, variables, and coefficients. Some questions to guide student thinking included:

- In your own words, what is a variable?
- What is an example of two like terms? Why?
- What is an example of unlike terms? Why?
- What is a coefficient?
- Can unlike terms have different coefficients? Explain.

The “big ideas” in this lesson were made explicit and conveyed to the teacher through a



discussion before she taught the next lesson and also in writing on the lesson plan. Based on what happened in this lesson, the next lesson was subsequently adjusted. This cycle of planning, teaching, observing, and debriefing occurred daily during the four weeks of the teaching experiment.

The following transcription is a whole class discussion about how the equal sign means “the same on both sides of an equation” or “balanced”. It exemplifies classroom discourse and provides insight into the motivations of the teacher. Moss knew from analyses of prior lessons and discussions with the teacher that many students in the class had an arithmetic view of the equal sign, where it only meant “to compute”. Thus, the goal of this lesson was to understand equivalent expressions and the idea of balancing a scale to solve for the unknown quantity in an equation. In this discussion, the students and teacher refer to a picture of a pan scale with different colored shapes that represent varying weights. The goal of this task is to balance the scale with shapes on either side.

Teacher: What is this?

John: Scale.

Teacher: What do we use scales for?

Gina: To measure.

Teacher: What is the goal?

Olivia: For it to be equal.

Teacher: What is another word for that?

Ian: Balanced.

Teacher: Balanced. Good. I have four shapes up there. A square, circle, triangle, and diamond. If I have a red square on this side. How do I make it balanced?

Class: Put a red square on the other side.

Teacher: If I add a blue circle or two blue circles, what do I have to do to the other side?

Class: Two blue circles.

Teacher: If I add a yellow diamond and another purple triangle, and another triangle, and a circle.

Class: Yellow diamond, purple triangle, purple triangle, and circle.

John: It's so easy.

Teacher: This shows that when we have a scale you want to make sure it is balanced. That is going to be the same concept when it comes to this third definition of variable. If I am given a problem like  $13 = x - 1$ , according to this rule, what do I have to

- do? Raise your hands.
- Jason: Solve for the unknown variable.
- Teacher: Solve for the unknown variable or solve for  $x$ . And I wrote  $x$  because on the standardized test most of the time the variable will be  $x$ . When I do this, I want to think of the scale and remember that both sides are balanced.

Following this lesson, Moss and the teacher determined that students seemed to understand the balancing concept of equations but needed to understand opposite arithmetic operations and mathematical symbols to solve for an unknown. Therefore, in the next lesson, students were still provided with a picture of a scale, but it had numbers and variables, instead of colored shapes.

### Retrospective analyses

In addition to the daily lesson logs, the data analyzed retrospectively in the design experiment consist of video recordings of classroom discourse and scanned copies of students' written work, including a pretest and posttest. As part of the articulation of an HLT following the conclusion of the teaching experiment, Moss developed a progression of the students' levels of thinking about expressions and equations (see Figure 4).

|                        |  |
|------------------------|--|
| Label Thinker          | Used letters to label a category or item (e.g. $c$ is cupcakes)  |
| Formulaic Thinker      | Used letters to keep a record of a quantity that has a feel of a known (e.g. Given a context where there are 2 girls and 8 boys, then $g + b = 10$ )   |
| Substituter            | Understood that a letter can be substituted for a given value (e.g. $g = 2$ , so $2 + b = 10$ )  |
| Solver                 | Understood that an equation can be solved for an unknown value (e.g. $2 + b = 10$ , so $b = 8$ ) and an expression can be simplified to find an unknown value (e.g. $4 + 3 = y$ , so $y = 7$ ) |
| Correspondence Thinker | Understood a letter as representing a changing quantity and that a relationship exists between inputs and outputs (e.g., in $y = x+3$ )  |

Figure 4. Levels of thinking about expressions and equations (Adapted from Moss, 2014)

The intent of this section is to share examples from student work that demonstrate each level of thinking about expressions and equations.

The task in Figure 5 was on the posttest. In this task, students are asked to find a missing number. The student's work on the task (Figure 5) demonstrates *label thinking*. She indicated

that a letter is a label for a known category, using  $s$  for songs. For example, she wrote the equation  $27s - 18s = 11$  and labeled  $s = \text{songs}$  to find the missing number of songs.

Figure 5. Student's work that shows *label* thinking

The letter  $s$  is used to label *songs* where  $27s$  is interpreted as 27 *songs* and  $18s$  is interpreted as 18 *songs*. The  $s$  in the student's work is not a quantity, but, rather, is labeling 27 songs and 18 songs.

The work in Figure 6 demonstrates *solver thinking*. The student uses the letter  $L$  to represent the missing number of songs and wrote the equation  $18 + L = 27$ . In this case,  $L$  is a yet-to-be-known quantity. In the levels of thinking, the student set up an equation and balanced the equation by subtracting 18 from both sides of the equal sign to find the missing quantity demonstrating solver level of thinking.

Figure 6. Student work that shows *solver* thinking

In another task, given on the first day of the instructional unit, students were asked to write in their notebooks an expression that shows the number of adults and the number of kids in

their respective families (see Figure 7).

5kids + 5Adults = 10 people

$S + I = X$

$S = \text{kids}$   
 $I = \text{Adults}$   
 $X = \text{Total members}$

Figure 7. Student's work that shows *formulaic* thinking

The work in Figure 7 shows the use of the letter  $S$  to keep a record of the number of *kids*, the letter  $I$  to keep a record of the number of *adults*, and the letter  $X$  to represent the *total*. The quantities in these cases are known and the student wrote an equation  $S + I = X$  to begin to understand that a letter represents a known quantity. This work is evidence of *formulaic thinking* because the student used letters to keep a record of given, known quantities.

On this same task, another student wrote the equation  $L + 2 = G$  (Figure 8). She substituted the letter  $L$  for the quantity 2 and the letter  $G$  for the quantity of 4. This student's thinking is an example of *substituter thinking*.

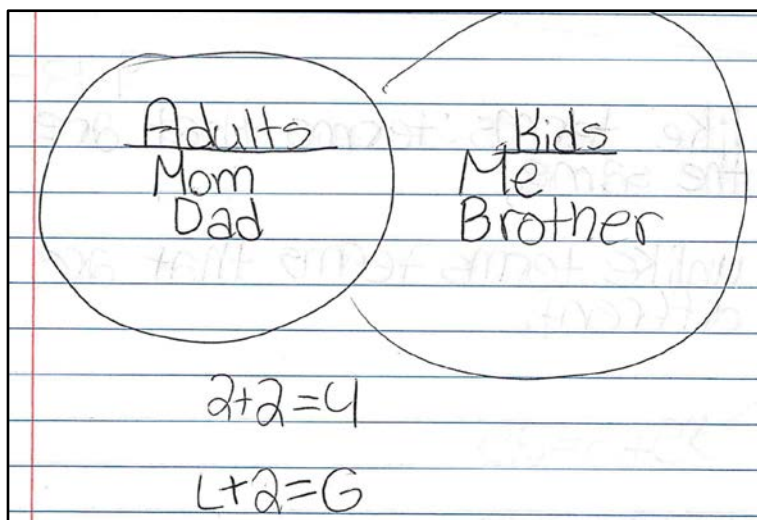


Figure 8. Student's work that shows *substituter* thinking.

*Substituters* use letters to make a one-to-one correspondence with a known quantity. This student

understood that the letters were assigned specific quantities and could be substituted into the equation. She reasoned that the given quantity was a *replacement* for the letter.

*Correspondence thinkers* begin to understand that a letter can represent a changing quantity, as opposed to a yet-to-be-known quantity and realize that there is a relationship between inputs and outputs. In Figure 9, a correspondence relationship is described and the student represents the relationship with an arrow diagram (table), algebraic equation, and graph.

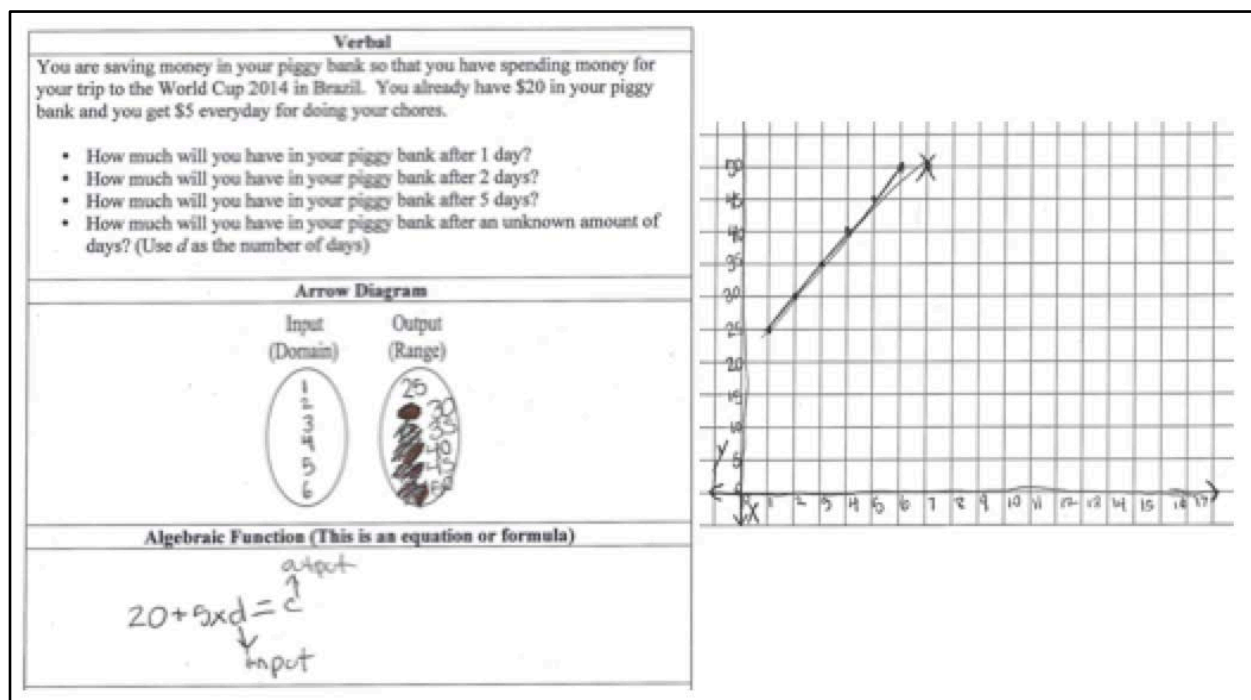


Figure 9. Student's work that depicts *correspondence* thinking

In the arrow diagram, the input and output are changing quantities that relate to one another. In the algebraic function, the variables  $d$  and  $c$  represent these changing quantities. The graph of the line also shows changing quantities with a representation of how the dollar amount in the piggy bank increases over time. The student labeled the input,  $d$ , and the output,  $c$ , in the equation. Functional thinkers continue to think of an equation as balanced and begin to see an equation as relating inputs and outputs.

The levels of thinking in Figure 4 are related because students must think of letters as known and yet-to-be-known values depending on the given algebraic situation. Figure 10 shows a relationship between thinking of a letter as a known value and of thinking of a letter as a yet-to-be-known value. As students began to think flexibly about the meaning of letters that represent numbers, they were able to engage in doing algebra as correspondence thinkers.

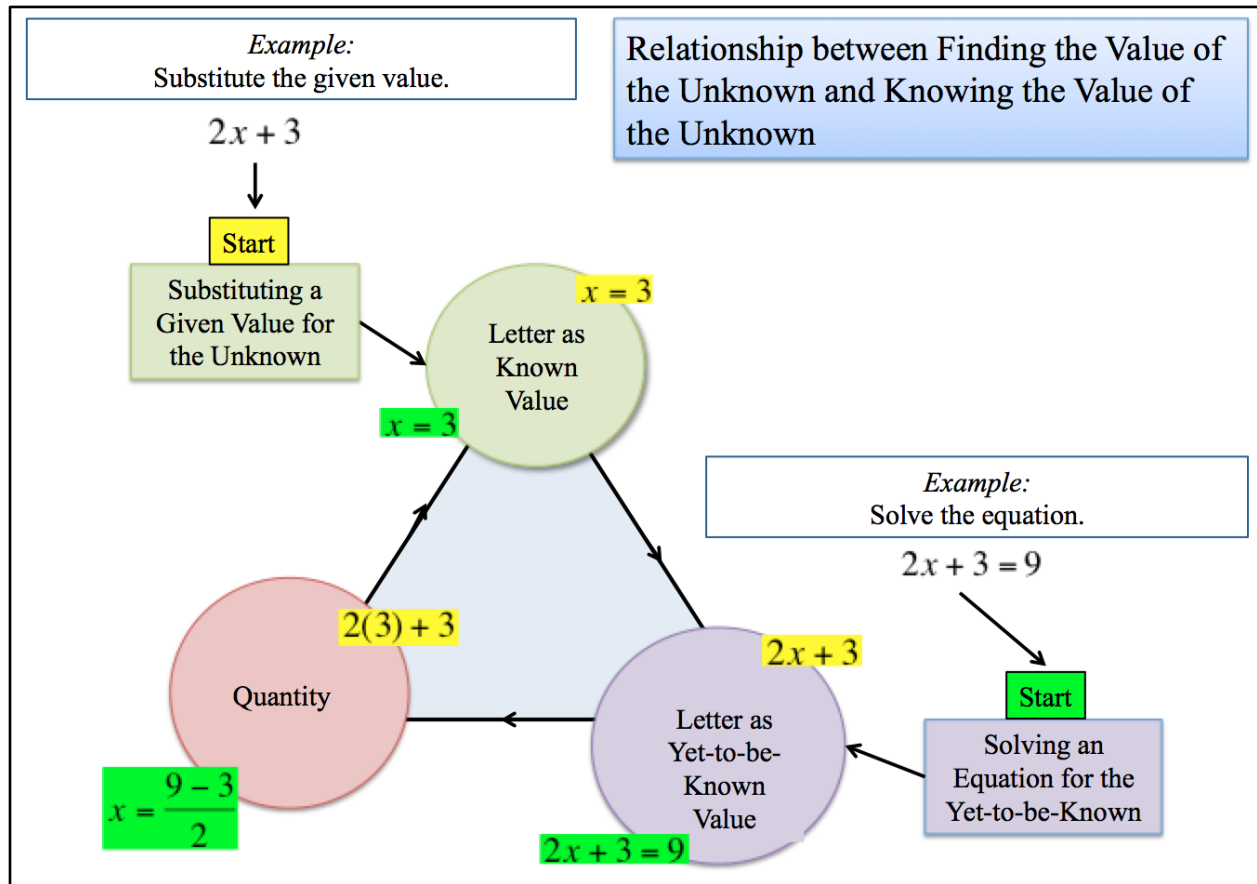


Figure 10. The relationship between thinking of letters as known and yet-to-be-known values.

### Analyses from the Re-emergent Perspective

As part of the design, Moss conducted four individual student interviews during the final week of the teaching experiment. The interviews lasted approximately 30 minutes, and they consisted of a sequence of written tasks followed by requests for verbal explanations. The four students who were interviewed—Cris, Enrique, Gina, and Maria (each pseudonyms)—were

seated together in the classroom throughout the teaching experiment and were consistently the focus of one of two video-cameras that recorded classroom discourse.

### Analyses of Interview Data

Boyce's initial goals centered on forming conjectures involving relationships between students' numerical, quantitative, and algebraic reasoning and the soon-to-be-seen classroom mathematical practices. We present the results of analyses from the re-emergent perspective by first providing some examples of Boyce's interpretations from the task-based interviews. We relate this to students' levels of thinking about equations (Figure 4) and then describe how we collaborated to retrospectively analyze the other data.

**Cris' and Maria's reasoning with missing numbers sentences.** Maria and Cris each successfully solved Missing Number Tasks (see Figure 11), and their verbal responses to Moss' requests to explain their thinking revealed differences in the ways their written representations reflected their thinking. Maria did not write anything to represent her reasoning until *after* she had completed computations mentally. She started with writing '-3' on the right side of the equation. She then subvocally said, "set it equal" and, after 10 seconds, wrote " $x = 22$ ". Lastly, she indicated addition of 10 and 22 on the left side of her equation. When the interviewer asked her to explain, Maria said, "I knew it had to be equal, so I did  $x$  equals 22 because 22 plus 10 equals 32." Despite further interviewer prompting, Maria was unable to explain how she arrived at " $x=22$ " or how subtracting 10 could be related to her procedure.

| What if it was $10 + x = 35 - 3$ ? How would you solve for $x$ ? |                |
|--|----------------|
| Maria's Response   | Cris' Response |

Figure 11. Maria's and Cris' responses to missing number tasks

Cris' representations of repeated subtractions were in tandem with his obtaining intermediate results. Unlike Maria, he represented his *process* of reversing addition on the other side of the '='. But when Cris finished his second subtraction, and the interviewer asked him how he could check his work, Cris expressed that he unsure how his final result (22) related to the original task. He was able to check that the original number sentence was true by substituting the '22' for  $x$ , but not until the teacher suggested that activity. The two students thus revealed different gaps in their *schemes*, or ways of assimilating and operating in service of a goal (von Glasersfeld, 1995). Maria was able to assimilate the task as an equivalent missing number task:  $10 + \underline{\quad} = 32$  without representing this as subtraction. Cris, in the process of representing the sequence of subtractions, lost that the goal was to find the value of  $x$ .

**Gina's and Enrique's reasoning with missing quantities.** Unlike Cris and Maria, Enrique's and Gina's interviews included tasks situated in context (it was later revealed that each connected to a World Cup Soccer theme). Gina was initially unsure how to respond to the Tickets task, which involves missing rates (see Figure 12).



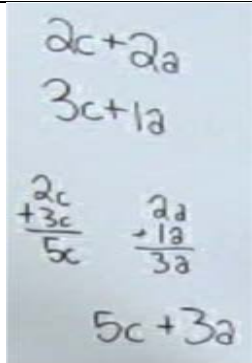
| Tickets Task   | Gina's Response   |
|--|---|
| <p>My family has 2 children and 2 adults. My friend, Jake's, family has 3 children and 1 adult. We don't know how much tickets cost for adults and how much tickets cost for children.</p> <p>Model this with an algebraic expression.</p> |  |

Figure 12. Gina's writing in response to the Tickets Task

The interviewer (Moss) suggested “just starting with how much adult tickets cost and how much children's tickets cost.” Gina used the letter ‘a’ for adult tickets and the letter ‘c’ for children's tickets in writing two expressions. Boyce noticed that Gina did not indicate understanding these letters as representing either unknown *rates* (dollars per ticket) or *quantities* (numbers of dollars). But after Moss asked Gina to represent “the total cost for both families,” Gina wrote the expression, “ $5c + 3a$ ”, suggesting that for her, the words “total cost” signaled a need to combine like terms (i.e.,  $2c + 3c$  and  $2a + 1a$ ).

Enrique completed a similar task in which rates were known and quantities were unknown. Enrique's justification for his response to the Equipment Task (Figure 13) was to explain “there's 40 dollars *of* cleats and 60 dollars *of* jerseys [emphasis added].” He also claimed that “you can put any letter as a variable, it doesn't matter.” Enrique thus indicated awareness of letters symbolizing unknown quantities, rather than labels. But, Enrique's response indicated he was not thinking about unknown unit *rates*, as he did not express 40 dollars *per* jersey. Later, when Enrique evaluated his expression for given values of the quantities of jerseys and cleats, he labeled the total cost with a ‘\$’ before performing the computation. Enrique thus used the ‘\$’ symbol as an abbreviation for a label when combining like terms in the same way Gina used

symbols ‘a’ and ‘c’.

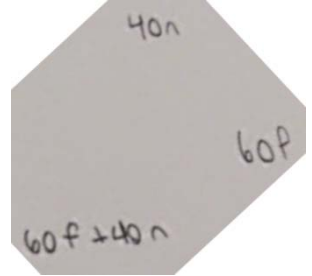
| Equipment Task  | Enrique's Response   |
|---|--|
| <p>The price of cleats is \$40 and the price of a jersey is \$60. I need to buy a certain number of cleats and a certain number of jerseys.</p> <p>Write an algebraic expression that shows this situation.</p> |  <p>The image shows a handwritten algebraic expression on a piece of paper. The expression is <math>40n + 60f</math>. The '40n' is written at the top, '60f' is written below it, and a plus sign is written between them. The paper is slightly tilted and has some other faint markings.</p> |

Figure 13. Enrique's writing in response to the Equipment Task

**Reasoning with whole numbers and fractions.** Enrique was the only student who was given an interview task involving fractions. He responded to the task, “Solve  $\frac{1}{2}x = 7$ ” by inverting the fraction to write the equation as “ $2/1 x = 7$ ”. He next re-wrote the equation as “ $2x = 7$ ”. Moss stopped him, and suggested that he instead divide both sides of the original equation by  $\frac{1}{2}$ . After writing  $\frac{1}{2} / \frac{1}{2} x = 7 / \frac{1}{2}$ , Enrique crossed out the  $\frac{1}{2} / \frac{1}{2}$  on the left side of the equation, replacing it first with  $1x$  and then  $x$ . On the right side, he said he was stuck because “7 divided by  $\frac{1}{2}$  would be 3...we can't cut it in half unless we make it a fraction.” Moss then guided him through the “invert and multiply” procedure for fractions division to yield the result of 14. He then reasoned that this was correct by confirming that half of 14 is indeed 7. Enrique knew that inverting the fraction was involved in the procedure to solve the equation, but (especially in contrast to confident responses in earlier tasks) he demonstrated that he did not how or why.

**Conjectures about students' participatory or anticipatory schemes.** Boyce conjectured from the interviews that Gina, Maria, and Cris had each constructed *participatory* schemes for understanding letters as labels and for understanding letters as representations for unknown (whole) numbers. The students' activities were participatory because they were connected to their reasoning, but they needed guidance or prompting from the teacher (Tzur, 2007). In contrast, Enrique's responses indicated he had interiorized these understandings —

they were *anticipatory* for him and did not require teacher assistance or scaffolding. It was striking that Enrique's responses to the tasks involving fractions were much like the other three students' responses to tasks involving whole numbers: he attempted arithmetic procedures in search of a meaningful result and appealed to Moss for guidance.

### Re-emergent Analyses of Classroom Learning Progression

After Boyce shared and discussed his interpretations of the interviews at the end of the teaching experiment with Moss, Moss revealed the unit plan, associated learning trajectory, and scope of data sources. In particular, Moss revealed that the class demonstrated growth as they moved from thinking of letters as abbreviations for qualities to substituting numbers for variables in algebraic expressions, as evidenced by their written work in their notebooks and on pre/post assessments. For example, Figure 14 shows the contrast in Maria's learning to substitute numerical values for letters in evaluating expressions between her pretest and posttest. Boyce learned that the trajectory Moss described referred to class' progression as a whole – she had not focused her analysis on progressions of individual students' learning.

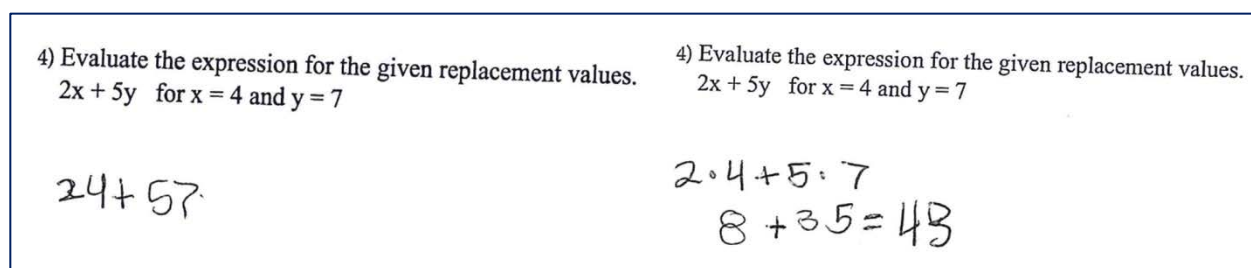


Figure 14. Maria's evaluating expressions on the pretest (left) and posttest (right)

We reviewed transcriptions and video-recordings of classroom activities and students' written work to understand the relationships between students' participation in classroom activities and their interview responses. Moss searched through her transcriptions of the classroom recordings and prior analyses to identify potentially important segments (e.g., those

with the word “fractions” or when expressions were first introduced) for Boyce to examine. We focused particularly on small-group interactions between Cris, Enrique, Gina, and Maria and their teacher to potentially corroborate or refute inferences from the interviews. Boyce watched these segments in reverse chronological order, meeting with Moss weekly to discuss and plan for the next (chronologically previous) video segments, to repeatedly form and test conjectures about how the students’ activities suggested they had constructed (procedural, participatory or anticipatory) schemes for reasoning algebraically.

**Classroom discourse.** The culturally relevant context of World Cup soccer had connected topics throughout the instructional unit. A typical lesson began with the teacher leading a whole-class discussion, proceeded to cooperative learning in small groups, and closed with another whole-class discussion. In the beginning of a lesson, the teacher would re-voice and represent students’ verbal contributions in response to open-ended questions about previous class activities or teacher-introduced concepts and definitions. Whole-class discussions at the close of sessions were student-centered, as the teacher would either call on students or ask for volunteers to come to the front of the room to present their work orally and visually. Enrique volunteered most often for this closing segment, and Gina also often volunteered during the whole-class discussion at the beginning of class. Cris and Maria were more likely to be called on than volunteer.

Of the four focus-group students, Enrique was the only student who verbalized understanding variables as unknown *quantities* during classroom discussion. That he was constructing participatory schemes for reasoning with unknown quantities when the others were constructing procedural schemes was evident in his leading small-group discussions. The other three students had needed guidance from the interviewer on how their arithmetic activity related

to solving or simplifying tasks in the interviews. In the classroom activities, this guidance had come from the teacher or from Enrique.

**Units coordination.** Boyce attended in particular to the students' *units coordination* (Norton, Boyce, Ulrich, & Phillips, 2015, see Figure 15) throughout his analysis. Units coordination has been implicated for students' understanding of fractions, integer arithmetic, and linear equations (Hackenberg & Lee, 2015; Ulrich, 2015; 2016).

| Stage | Units Coordination Description  |
|-------|---|
| 1     | Children require activity to form a numerical composite (pre-fractional)  |
| 2     | Children can assimilate with a unit composed of other units and further (de)compose units in activity (can reason with proper fractions). |
| 3     | Children can assimilate with units within units within units (can reason with improper fractions as numbers).                             |

Figure 15. Stages of units coordinating development (Adapted from Norton et al., 2015)

Boyce's inferences regarding Enrique's reasoning with fractions during the interview were corroborated by Enrique's indicating similar perturbation during the whole-class and small-group sessions involving fractions. In the classroom activities, none of the four students demonstrated understanding of (improper) fractions as numbers<sup>1</sup> (Hackenberg, 2010). The other three students did not verbalize reasoning with fractions in whole-class discussion at all. The data suggests that Enrique was at Stage 2 of units coordination, as reasoning with unknown rates and improper fractions are both in the purview of Stage 3 students (Hackenberg, & Lee, 2015), and that the other three students were at Stage 1 of units coordination. However, there was limited data to support these inferences because the students were not specifically tasked with communicating their arithmetic reasoning.

**Comparisons of students' written work.** Each student was responsible for keeping a

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<sup>1</sup> One of the 22 students in the class demonstrated this understanding, but he was not part of the interview study.

notebook with his or her work on class activities and a copy of what had been written on the board by the teacher. For example, the three meanings of variables written in Cris' notebook were written, essentially verbatim, in each of the students' notebooks (see Figure 9). These meanings had been introduced over the course of weeks and had successively become "taken-as-shared." The four students' work in response to other prompts, such as, "write an expression for the number of sides of hexagon and a pentagon" included differences suggesting individuals' autonomy, but the videos indicated that the other students often turned to Enrique to verbally explain his response before completing or revising their own work.

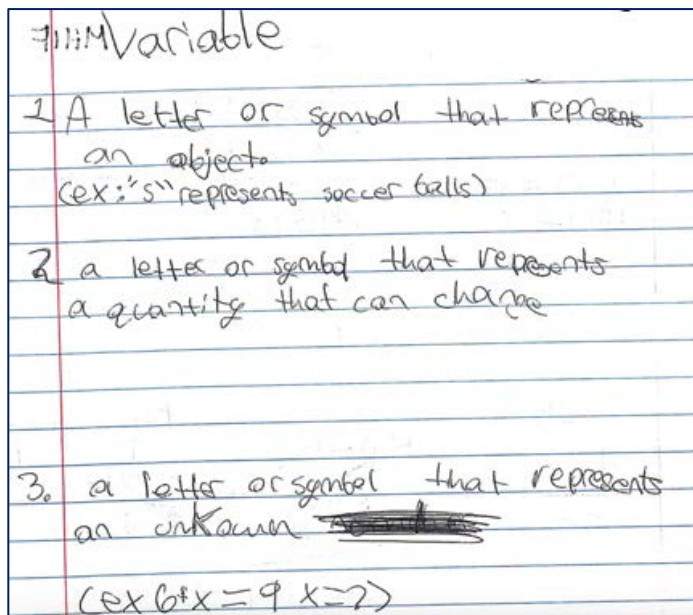


Figure 16. Cris' writing of class' taken-as-shared meanings of variables

The differences between Enrique's reasoning with variables as unknowns and the other three students' were particularly evident from one question on the post-test. The students were asked, "How many pairs of numbers can you find that add to 10? Express the number 10 as a sum of two numbers using variables." Figure 17 displays the four students' responses. Each of the students' interpretations involved writing arithmetic expressions, but only Enrique represented his arithmetic reasoning using variables as unknown numbers. This nuance was not noted prior

to the retrospective analysis, as Moss had noted that Cris' and Maria's responses were not appropriate but had given full credit for Gina's response.

|         |  |
|---------|--|
|         | Post-test response to "How many pairs of numbers can you find that add to 10? Express the number 10 as a sum of two numbers using variables."                    |
| Cris    | $2,5$ $10,1$<br>$5,2$ $1,10$   |
| Enrique | <del>10 pairs</del> <del>10</del> <del>10</del> <del>10</del> <del>10</del><br><del>10</del> <del>10</del> <del>10</del> <del>10</del> <del>10</del><br>$a+b=10$ |
| Gina    | $5+5=10$ $4+6=10$ $2x+8x=10x$<br>$1+9=10$ <del>10</del><br>$2+8=10$ $10+0=10$<br>$3+7=10$  |
| Maria   | $6+x=10$<br>$20-b=10$  |

Figure 17. Contrasting students' use of variables on the post-test

Another question on the post-test exemplifies how the four students' understanding of procedures for solving algebraic equations followed a different pattern. Though all four students arrived at the correct solution ' $x=2$ ' for the equation ' $2x + 10 = 14$ ', only Gina represented inverting both addition and multiplication to arrive at the solution (see Figure 18). Cris had started writing subtraction of 10, but then paradoxically indicated dividing the '10' by 2. He

crossed this out and then responded similarly to Maria, who included only an evaluation at  $x=2$ . Thus, neither Cris nor Maria indicated reasoning with reversing arithmetic operations to solve the equation. Enrique's representing inverting addition but not multiplication suggests his justification of the solution was participatory, rather than procedural, whereas Gina's representation precisely mirrored the teacher-demonstrated procedure for solving two-step equations.

17) Solve the equation.  
 $2x + 10 = 14$   
 $2x + 10 - 10 = 14 - 10$   
 $2x + 0 = 4$   
 $2x = 4$   
 $\div 2 \downarrow \div 2$   
 $1 \cdot x = 2$

17) Solve the equation.  
 $2x + 10 = 14$   
 $2 \cdot 2 = 4 + 10 = 14$   
 $x = 2$

$\frac{14}{-10} = 2 \cdot 2$

$x = 2$

Figure 18. Gina's (left) and Enrique's (right) post-test equation-solving

## Discussion

Our collaborative analysis revealed individual-level differences in the classroom-level learning trajectory that had not been apparent during the emergent analysis. Differences in students' ways of participating in classroom activities and their reasoning about variables as unknown quantities in the interviews were associated with their mental (units coordinating) structures. But because students' arithmetic and quantitative reasoning were not a focus of analysis until after data collection was complete, the data from which students' units coordination could be analyzed was limited. With that caveat, the findings are consistent with research that suggests that many U.S. students entering sixth-grade coordinate fewer than three levels of units (Boyce & Norton, 2016), and that such differences affect the types of schemes



students might construct to reason about algebraic expressions and equations (Hackenberg & Lee, 2015). In addition to difficulties with fractions concepts, the class was unfamiliar with arithmetic operations involving negative integers – the teacher realized this and adjusted the class activities to exclude equations with negative integers; hence they had not appeared in the student interviews. These results suggest it may be better to integrate early algebra objectives with other sixth-grade learning goals to support students’ development of participatory, rather than procedural, early algebraic understandings, or to continue to revisit algebraic goals throughout the school year.

## Conclusions

In this paper, we introduced the notion of *re-emergent perspective* to characterize a collaborative approach to retrospective analysis. Moss’ analyses of students’ reasoning during the design experiment were focused on how students’ understandings were (or were not) compatible with the taken-as-shared understandings about variables underlying the emerging hypothetical learning trajectory. Boyce’s focusing first on four students’ development at the conclusion of the teaching experiment without prior knowledge of the hypothetical trajectory was powerful for distinguishing what was procedural and what was participatory about that shared understanding. Moreover, the re-emergent analyses helped to provide additional causal mechanisms for differences in learning outcomes within the hypothetical learning trajectory. Our results thus exemplify how retrospective analyses from researchers adopting different perspectives might inform our practice as mathematics education researchers.

As the researcher situated in a re-emergent perspective begins with limited exposure to the context of a design experiment, the relationships between his or her theoretical perspective, experiences, and goals and those of the researcher immersed in the data are paramount. We have

discussed our approach focusing on connections between classroom mathematical practices and individuals' mathematical reasoning, in which individual interviews at the end of a classroom design experiment were a starting point for re-emergent analysis. Although a weakness of our approach to retrospective analysis is the appropriateness of analyzing data for an unanticipated purpose, the independence of data collection and data analysis was also a strength. Researchers might also conduct re-emergent analyses focused on other aspects within the interpretative framework, such as connections between students' beliefs and socio-mathematical norms, that could lend themselves to similar approaches to continue to develop and refine mathematics learning trajectories in design research.

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